Insurance: Mathematics and Economics 65 (2015) 55–65

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

A risk model with renewal shot-noise Cox process

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ARTICLE INFO

Article history:
Received October 2014
Received in revised form
July 2015
Accepted 21 August 2015
Available online 3 September 2015

JEL classification:
G22
C10
C60

MSC:
primary 91B30
secondary 60J75
65C05

Keywords:
Risk model
Ruin probability
Renewal shot-noise Cox process
Piecewise-deterministic Markov process
Martingale method
Importance sampling
Change of probability measure
Rare-event simulation

ABSTRACT

In this paper we generalise the risk models beyond the ordinary framework of affine processes or Markov processes and study a risk process where the claim arrivals are driven by a Cox process with renewal shot-noise intensity. The upper bounds of the finite-horizon and infinite-horizon ruin probabilities are investigated and an efficient and exact Monte Carlo simulation algorithm for this new process is developed. A more efficient estimation method for the infinite-horizon ruin probability based on importance sampling via a suitable change of probability measure is also provided; illustrative numerical examples are also provided.

1. Introduction

In insurance modelling a *Poisson process* has a long history of being used as a classical model for the claim-arrival process. Extensive discussions from both applied and theoretical viewpoints can be found in early literature, Cramér (1930), Cox and Lewis (1966), Bühlmann (1970) and Çınlar (1974). A Poisson process is a simple counting process that measures the number of claim occurrences within a period of time. It is easy to use mainly due to its memoryless property. However, the exponential distribution underlying claim-arrival times is often not appropriate to use for modelling the interarrival times of claim arrivals in real situations. The likelihood of a claim given the time elapsed since the previous one is not necessarily constant throughout time. There has been a significant volume of literature that questions the appropriateness of the Poisson process in insurance modelling, in particular for catastrophic events; see Seal (1983) and Beard et al. (1984).

As an alternative point process to generate claim arrivals we can employ a *non-homogeneous Poisson process* or a *Cox process* first introduced by Cox (1955b). A Cox process is a natural generalisation of a Poisson process by considering the intensity of Poisson process as a realisation of a random measure (Møller, 2003). The Cox process provides the flexibility of letting the intensity not only depend on time but also allowing it to be a stochastic process. Hence, it can be viewed as a two-step randomisation procedure which can deal with the stochastic nature of catastrophic loss occurrences in the real world.

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http://dx.doi.org/10.1016/j.insmatheco.2015.08.009
0167-6687/

Moreover, *shot-noise processes* (Cox and Isham, 1980) are particularly useful to model claim arrivals; they provide measures for frequency, magnitude and the time period needed to determine the effect of catastrophic events within the same framework; as time passes, the shot-noise process decreases as more and more losses are settled, and this decrease continues until another event occurs which will result in a positive jump. Therefore, the shot-noise process can be used as the intensity of a Cox process to measure the number of catastrophic losses. Previous works on insurance applications using a shot-noise process or a *Cox process with shot-noise intensity* can be found in Klüppelberg and Mikosch (1995), Brémaud (2000), Dassios and Jang (2003), Jang and Krvavych (2004), Torrisi (2004), Dassios and Jang (2005), Albrecher and Asmussen (2006), Macci and Torrisi (2011), Zhu (2013) and Schmidt (2014).

In reality, when catastrophic events occur, the arrivals of the associated claims arising from them could also depend on the time elapsed since the previous catastrophic events (e.g. floods, storms, hails, bushfires, earthquakes and terrorist attacks). Hence, the information provided by the time intervals between the primary events is also valuable in insurance. To model the arrivals of claims arising from catastrophic events where the interarrival times between the primary events are additionally included, further improved models are required. For this purpose, in this paper we introduce a shot-noise process driven by an *ordinary renewal process* as the claim-arrival intensity process. It is a Cox process that further generalises the risk models beyond the ordinary framework of affine processes or Markov processes.

The paper is structured as follows. Our model of the Cox process with renewal shot-noise intensity is introduced and the mathematical definition is provided in Section 2. This process is then used as the claim-arrival process in a risk model, and we find an appropriate martingale in Section 3 to find the upper bounds of the finite-horizon and infinite-horizon ruin probabilities in Section 4. In Section 5, we develop an associated numerical algorithm for simulating this new risk process, and it is used to estimate the ruin probabilities based on crude Monte Carlo simulation. A more efficient estimation method for the infinite-horizon ruin probability based on importance sampling is also provided. To illustrate in detail how this proposed model can be implemented, we provide related numerical examples in Section 6. There, we specify that both the claim sizes and jump sizes in the claim-arrival intensity follow exponential distributions and the interarrival times follow an inverse Gaussian distribution.

2. A renewal shot-noise cox process

We generalise the classical *Cox process with Poisson shot-noise intensity* to a *Cox process with renewal shot-noise intensity* as defined below. The arrivals of jumps follow a renewal process and the impact of each jump decays exponentially over time.

Definition 2.1 (*Renewal Shot-noise Cox Process*)**.** A *renewal shot-noise Cox process* (*Cox process with renewal shot-noise intensity*) is a point process $N_t \equiv \{T_j\}_{j=1,2,\ldots}$ on $\mathbb{R}_+$ with renewal shot-noise intensity λ_t, i.e. a non-negative shot-noise process driven by an ordinary renewal process specified by

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i=1}^{M_t} Y_i e^{-\delta\left(t-T_i^*\right)}, \quad t \geq 0,$$

where

- λ_0 is the initial intensity;
- $\delta > 0$ is the constant rate of exponential decay;

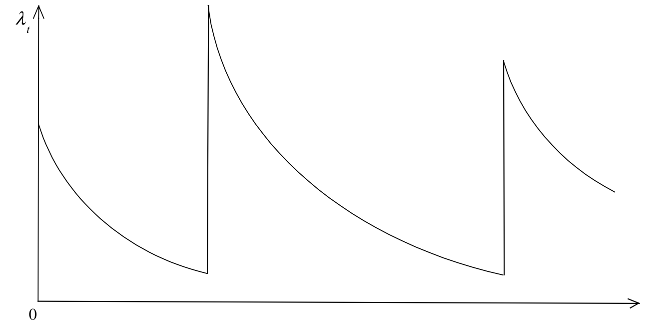

Fig. 1. A sample path of renewal shot-noise intensity process λ_t.

- $\{M_t\}_{t\geq 0}$ is a *renewal process* with arrival times $\{T_i^*\}_{i=1,2,\ldots}$, i.e. $M_t \equiv \{T_i^*\}_{i=1,2,\ldots}$;
- $\{Y_i\}_{i=1,2,\ldots}$ is a sequence of *i.i.d.* random variables (sizes of renewal jumps or shots) with distribution function $H(y)$, $y > 0$, which is assumed to be absolutely continuous with density function $h(y)$ and independent of M_t.

A sample path of the renewal shot-noise intensity process λ_t is illustrated in Fig. 1. If M_t is a Poisson process instead, then λ_t is a classical *shot-noise process* (Cox and Isham, 1980). If we set $Y_i \equiv 1, \lambda_0 = 0$ and replace M_t by the point process N_t itself, then N_t is the classical Markovian self-exciting *Hawkes process* (Hawkes, 1971) on the half line. In this paper, we assume that M_t follows a renewal process, and our process is then a special case of *generalised shot-noise Cox processes* (Møller and Torrisi, 2005).

Some distributional properties of this process such as moments have been summarised in Dassios and Jang (2012). Note that this process is no longer within the usual framework of an affine process (Duffie et al., 2000) or a Markov process due to the additional renewal components. In order to establish a Markovian framework, we need to include a supplementary variable U_t, *the time elapsed since the last jump arrived* in the intensity process λ_t, i.e.

$$U_t := t - \sum_{i=1}^{M_t} R_i,$$

where $\{R_i\}_{i=1,2,\ldots}$ are the interarrival times of the renewal process M_t, i.e.

$$R_i := T_i^* - T_{i-1}^*, \quad i = 1, 2, \ldots, \; T_0^* = 0,$$

and they are *i.i.d.* with distribution function $P(u)$, $u > 0$, which is assumed to be absolutely continuous with density function $p(u)$. The idea of adding this supplementary variable U_t to make the process Markovian can be found as early as in Cox (1955a). U_t increases at unit rate till a jump arrives; then it goes back to 0. Note that, if $\rho(u)$ is the failure rate of the distribution, we have

$$P(u) = 1 - \exp\left(-\int_0^u \rho(v)\,\mathrm{d}v\right),$$

$$p(u) = \rho(u)\exp\left(-\int_0^u \rho(v)\,\mathrm{d}v\right),$$

and $\rho(u) = \frac{p(u)}{\bar{P}(u)}$ where $\bar{P}(u) := 1 - P(u)$.

3. A risk process driven by a renewal shot-noise Cox process

Now, let us consider an insurance company with surplus process X_t in continuous time on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume

$$X_t = X_0 + ct - \sum_{j=1}^{N_t} Z_j, \quad t \geq 0, \tag{1}$$

where

- $X_0 \geq 0$ is the initial reserve at time $t = 0$;
- $c > 0$ is the constant rate of premium income;
- N_t is a renewal shot-noise Cox process (defined by Definition 2.1) with associated claim-arrival times $\{T_j\}_{j=1,2,\ldots}$;
- $\{Z_j\}_{j=1,2,\ldots}$ are claim sizes which are assumed to be *i.i.d.* with distribution function $Z(z), z > 0$. We also assume they are independent of N_t.

The generator of the joint process (X_t, λ_t, U_t, t) acting on a function $f(x, \lambda, u, t)$ belonging to its domain is given by

$$\mathcal{A} f(x, \lambda, u, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} - \delta\lambda \frac{\partial f}{\partial \lambda} + c\frac{\partial f}{\partial x} + \lambda \left(\int_0^\infty f(x - z, \lambda, u, t)\, \mathrm{d}Z(z) - f(x, \lambda, u, t) \right) + \frac{p(u)}{\bar{P}(u)} \left(\int_0^\infty f(x, \lambda + y, 0, t)\, \mathrm{d}H(y) - f(x, \lambda, u, t) \right), \tag{2}$$

where $f : (-\infty, \infty) \times (0, \infty) \times (0, \infty) \times \mathbb{R}^+ \to (0, \infty)$. It is sufficient that $f(x, \lambda, u, t)$ is differentiable w.r.t. x, λ, u, t for all x, λ, u, t and that

$$\left| \int_0^\infty f(x - z, \cdot, \cdot, \cdot)\, \mathrm{d}Z(z) - f(x, \cdot, \cdot, \cdot) \right| < \infty$$

for $f(x, \lambda, u, t)$ to belong to the domain of the generator $\mathcal{A}$. For details on generators of *piecewise deterministic Markov processes* we refer to Davis (1984), Dassios and Embrechts (1989), Davis (1993) and Rolski et al. (2008).

For simplicity, we denote first-order moments by

$$\pi_1 := \int_0^\infty u p(u) \mathrm{d}u, \qquad \alpha_1 := \int_0^\infty y \mathrm{d}H(y),$$
$$\gamma_1 := \int_0^\infty z \mathrm{d}Z(z).$$

We also denote the Laplace transforms the moment generating functions by

$$\hat{p}(\nu) := \int_0^\infty e^{-\nu u} p(u) \mathrm{d}u, \qquad \hat{h}(\nu) := \int_0^\infty e^{-\nu y} \mathrm{d}H(y),$$
$$\phi(\nu) := \int_0^\infty e^{\nu z} \mathrm{d}Z(z).$$

We will be assuming existence of the above where necessary.

Lemma 3.1. *The net profit condition under the probability measure $\mathbb{P}$ is*

$$c > \frac{\gamma_1 \alpha_1}{\delta \pi_1}. \tag{3}$$

Proof. If the net profit condition holds, then, the expected premium received between two successive claims should exceed the expected amount of a claim loss, i.e. $c\mathbb{E}[T'] > \mathbb{E}[Z_j]$ where T' is the interarrival time of loss claims. It is also equivalent to the condition

$$\frac{\mathrm{d}}{\mathrm{d}\nu} \left[\hat{p}(c\nu) \hat{h}\left(-\frac{\phi(\nu) - 1}{\delta} \right) \right] \bigg|_{\nu=0} < 0. \quad \square$$

Lemma 3.2. *Consider the equation*

$$\hat{p}(\theta + c\nu)\, \hat{h}\left(-\frac{\phi(\nu) - 1}{\delta} \right) = 1, \tag{4}$$

for a constant $\theta \geq 0$. Then, the following are true:

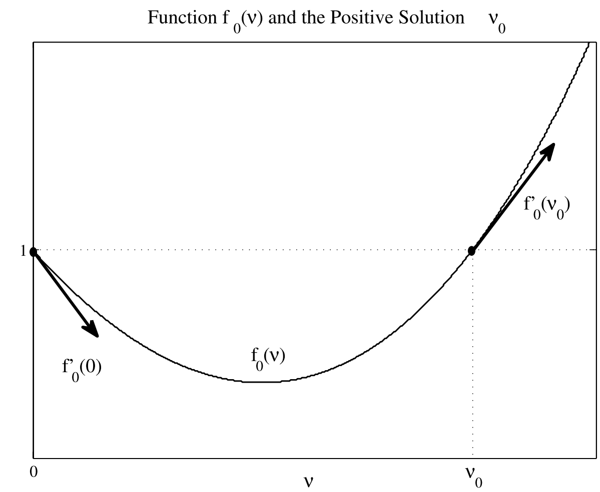

Fig. 2. Function $f_0(\nu)$ and the positive solution ν_0.

(i) *for $\theta > 0$, there exists a unique positive ν_θ such that (4) is satisfied for $\nu = \nu_\theta$;*
(ii) *in particular, for $\theta = 0$, under the net profit condition (3), there exists a unique positive ν_0 such that (4) is satisfied for $\nu = \nu_0$.*

Proof. Define

$$f_\theta(\nu) := \hat{p}(\theta + c\nu)\hat{h}\left(-\frac{\phi(\nu) - 1}{\delta} \right), \quad \theta \geq 0, \tag{5}$$

which is a convex function of ν for all $\theta \geq 0$, as its second derivative w.r.t. ν is given by

$$f_\theta''(\nu) = \int_{y=0}^\infty \int_{u=0}^\infty \left[\left(-cu + \frac{\phi'(\nu)}{\delta} y \right)^2 + \frac{\phi''(\nu)}{\delta} \right] \times e^{-(\theta + c\nu)u} e^{\left(\frac{\phi(\nu)-1}{\delta}\right)y} p(u) \mathrm{d}u \mathrm{d}H(y) \geq 0.$$

Also at $\nu = 0$, we have

$$f_\theta(0) = \hat{p}(\theta + c\nu)\hat{h}\left(-\frac{\phi(\nu) - 1}{\delta} \right) = \hat{p}(\theta) < 1.$$

Hence, if $\theta > 0$, there exists a unique ν_θ which is positive and satisfies (4).

In particular, for $\theta = 0$, we need the first derivative to be negative at $\nu = 0$ in order for ν_0 to exist, where the uniqueness is guaranteed by convexity. The derivative at $\nu = 0$ is

$$f_0'(\nu)\big|_{\nu=0} = -c\pi_1 + \gamma_1 \frac{\alpha_1}{\delta},$$

and this is negative by (3), also see Fig. 2. $\square$

Using Lemma 3.2, we will now find a suitable martingale which will be used to derive the upper bounds of the infinite-horizon and finite-horizon ruin probabilities in Section 4.

Theorem 3.1. *Suppose the net profit condition (3) holds. In this case,*

$$e^{-\nu_\theta X_t} e^{-\theta t} e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_t} \frac{\int_{U_t}^\infty e^{-(\theta + c\nu_\theta)\nu} p(\nu) \mathrm{d}\nu}{e^{-(\theta + c\nu_\theta)U_t} \bar{P}(U_t)} \tag{6}$$

is a $\mathbb{P}$-martingale.

Proof. From (2), $f(x, \lambda, u, t)$ has to satisfy the condition $\mathcal{A}f = 0$ for it to be a martingale. Setting

$$f(x, \lambda, u, t) = e^{-\nu x} e^{-\theta t} e^{\frac{\phi(\nu)-1}{\delta}\lambda} \hbar(u)$$

in (2), we get the equation

$$\hbar'(u)-(\theta+c\nu)\,\hbar(u)+\frac{p(u)}{\bar{P}(u)}\left[\hat{h}\left(-\frac{\phi(\nu)-1}{\delta}\right)\hbar(0)-\hbar(u)\right]=0. \tag{7}$$

Solving (7), we have

$$\hbar(u)=\hbar(0)\,\frac{\int_u^\infty e^{-(\theta+c\nu)v}p(v)\mathrm{d}v}{e^{-(\theta+c\nu)u}\bar{P}(u)}\hat{h}\left(-\frac{\phi(\nu)-1}{\delta}\right)+\hbar(0)\frac{1-\hat{p}(\theta+c\nu)\,\hat{h}\left(-\frac{\phi(\nu)-1}{\delta}\right)}{e^{-(\theta+c\nu)u}\bar{P}(u)}.$$

As the first term is bounded, for this function to belong to the domain of the generator the second term, which has infinite expectation should vanish. Hence, we set

$$\hat{p}(\theta+c\nu)\,\hat{h}\left(-\frac{\phi(\nu)-1}{\delta}\right)=1,$$

and therefore $\nu=\nu_\theta$. We now have

$$\hbar(u)=\hbar(0)\frac{\int_u^\infty e^{-(\theta+c\nu_\theta)v}p(v)\mathrm{d}v}{e^{-(\theta+c\nu_\theta)u}\bar{P}(u)}\hat{h}\left(-\frac{\phi(\nu)-1}{\delta}\right),$$

and the theorem is proved. □

4. Ruin probabilities

In this section, we obtain upper bounds for ruin probabilities, by employing a martingale approach. Similar ideas can be found in Dassios and Embrechts (1989), Dassios and Jang (2003) and Dassios and Zhao (2011). We define the *ruin time* by

$$\tau^*:=\inf\{t:X_t<0\}.$$

If $X_t\geq 0$ for all $t>0$, then, $\tau^*=\infty$. With the help of Theorem 3.1, we can obtain upper bounds for the finite-horizon ruin probability $\Pr\{\tau^*\leq T\mid X_0,\lambda_0,U_0\}$ for a fixed time $T>0$ and the infinite-horizon (ultimate) ruin probability $\Pr\{\tau^*<\infty\mid X_0,\lambda_0,U_0\}$. Numerical examples will be provided later in Section 6.

Theorem 4.1. *Suppose the net profit condition (3) holds. We then have*

$$\Pr\left\{\tau^*\leq T\mid X_0,\lambda_0,U_0\right\}\leq\inf_{\theta>0}\left\{\frac{\aleph(U_0,\theta)}{\underline{\aleph}(\theta)}e^{\theta T}e^{-\nu_\theta X_0}e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_0}\right\}, \tag{8}$$

$$\Pr\left\{\tau^*<\infty\mid X_0,\lambda_0,U_0\right\}\leq\frac{\aleph(U_0,0)}{\underline{\aleph}(0)}e^{-\nu_0X_0}e^{\frac{\phi(\nu_0)-1}{\delta}\lambda_0}, \tag{9}$$

where

$$\aleph(u,\theta):=\frac{\int_u^\infty e^{-(\theta+c\nu_\theta)v}p(v)\mathrm{d}v}{e^{-(\theta+c\nu_\theta)u}\bar{P}(u)},\qquad\underline{\aleph}(\theta):=\inf_{u>0}\left\{\aleph(u,\theta)\right\}. \tag{10}$$

Proof. Since (6) is a martingale and $\tau^*\wedge T:=\min\{\tau^*,T\}$ is a stopping time, by the *Optional Stopping Theorem*, we have

$$\mathbb{E}\left[e^{-\nu_\theta X_{\tau^*\wedge T}}e^{-\theta(\tau^*\wedge T)}e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_{\tau^*\wedge T}}\aleph(U_{\tau^*\wedge T},\theta)\,\middle|\,X_0,\lambda_0,U_0\right]=e^{-\nu_\theta X_0}e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_0}\aleph(U_0,\theta)$$

and therefore

$$\begin{aligned}&\mathbb{E}\left[e^{-\nu_\theta X_{\tau^*}}e^{-\theta\tau^*}e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_{\tau^*}}\aleph(U_{\tau^*},\theta)\,\middle|\,X_0,\lambda_0,U_0,\tau^*\leq T\right]\\&\times\Pr\left\{\tau^*\leq T\mid X_0,\lambda_0,U_0\right\}\\&+\mathbb{E}\left[e^{-\nu_\theta X_T}e^{-\theta T}e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_T}\aleph(U_T,\theta)\,\middle|\,X_0,\lambda_0,U_0,\tau^*>T\right]\\&\times\Pr\left\{\tau^*>T\mid X_0,\lambda_0,U_0\right\}\\&=e^{-\nu_\theta X_0}e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_0}\aleph(U_0,\theta).\end{aligned} \tag{11}$$

Hence, we have

$$\begin{aligned}&e^{-\nu_\theta X_0}e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_0}\aleph(U_0,\theta)\\&\geq\mathbb{E}\left[e^{-\nu_\theta X_{\tau^*}}e^{-\theta\tau^*}e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_{\tau^*}}\aleph(U_{\tau^*},\theta)\,\middle|\,X_0,\lambda_0,U_0,\tau^*\leq T\right]\\&\times\Pr\left\{\tau^*\leq T\mid X_0,\lambda_0,U_0\right\}.\end{aligned}$$

As $\tau^*\leq T$, we have

$$e^{-\theta\tau^*}\geq e^{-\theta T},\qquad e^{-\nu_\theta X_{\tau^*}}\geq 1,\qquad e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_{\tau^*}}\geq 1,$$
$$\aleph(U_{\tau^*},\theta)\geq\underline{\aleph}(\theta),$$

almost surely and

$$\Pr\left\{\tau^*\leq T\mid X_0,\lambda_0,U_0\right\}\leq\frac{\aleph(U_0,\theta)}{\underline{\aleph}(\theta)}e^{\theta T}e^{-\nu_\theta X_0}e^{\frac{\phi(\nu_\theta)-1}{\delta}\lambda_0},\quad\forall\,\theta\geq 0. \tag{12}$$

Hence, (8) follows. If we set $\theta=0$ in (12), we have (9) which is true for any time T. □

Remark 4.1. In order to investigate the monotonicity for the function $\aleph(u,\theta)$ of (10) w.r.t. the variable u, we calculate its first derivative

$$\frac{\partial}{\partial u}\aleph(u,\theta)=-\rho(u)+\frac{(\theta+c\nu_\theta)\,\bar{P}(u)+p(u)}{\left[\bar{P}(u)\right]^2}e^{(\theta+c\nu_\theta)u}\int_u^\infty e^{-(\theta+c\nu_\theta)v}p(v)\mathrm{d}v.$$

We observe that for any $z>0$, we have

$$\frac{\bar{P}(u+z)}{\bar{P}(u)}=\exp\left(-\int_u^{u+z}\rho(v)\mathrm{d}v\right)=\exp\left(-\int_0^z\rho(s+u)\mathrm{d}s\right).$$

We then observe that

- the failure rate $\rho(u)$ is a non-decreasing function of u, if and only if $\frac{\bar{P}(u+z)}{\bar{P}(u)}$ is a non-increasing function of u for any $z>0$;
- the failure rate $\rho(u)$ is a non-increasing function of u, if and only if $\frac{\bar{P}(u+z)}{\bar{P}(u)}$ is a non-decreasing function of u for any $z>0$.

We now rewrite $\aleph(u,\theta)$ as

$$\begin{aligned}\aleph(u,\theta)&=\frac{\int_u^\infty e^{-(\theta+c\nu_\theta)(v-u)}p(v)\mathrm{d}v}{\bar{P}(u)}\\&=\frac{\int_0^\infty e^{-(\theta+c\nu_\theta)s}p(u+s)\mathrm{d}s}{\bar{P}(u)}\\&=\frac{\int_0^\infty\left[1-\int_0^s(\theta+c\nu_\theta)\,e^{-(\theta+c\nu_\theta)z}\mathrm{d}z\right]p(u+s)\mathrm{d}s}{\bar{P}(u)}\\&=\frac{\bar{P}(u)-\int_{s=0}^\infty\int_{z=0}^s(\theta+c\nu_\theta)\,e^{-(\theta+c\nu_\theta)z}p(u+s)\mathrm{d}s\mathrm{d}z}{\bar{P}(u)}\end{aligned}$$

$$= \frac{\bar{P}(u) - \int_{z=0}^{\infty}\int_{s=z}^{\infty}(\theta + c\nu_\theta)\, e^{-(\theta+c\nu_\theta)z} p(u+s)\mathrm{d}s\mathrm{d}z}{\bar{P}(u)}$$

$$= 1 - (\theta + c\nu_\theta)\int_0^\infty e^{-(\theta+c\nu_\theta)z}\frac{\bar{P}(u+z)}{\bar{P}(u)}\mathrm{d}z.$$

Hence,

- if $\rho(u)$ is a non-decreasing function of u, then $\aleph(u,\theta)$ is a non-decreasing function of u and its minimum value is

$$\underline{\aleph}(\theta) = \aleph(0,\theta) = \hat{p}\left(\theta + c\nu_\theta\right); \tag{13}$$

- if $\rho(u)$ is a non-increasing function of u, then $\aleph(u,\theta)$ is a non-increasing function of u and its minimum value is

$$\underline{\aleph}(\theta) = \aleph(\infty,\theta) = \lim_{u\to\infty}\frac{p(u)}{p(u) + (\theta + c\nu_\theta)\bar{P}(u)}$$

$$= \lim_{u\to\infty}\frac{\rho(u)}{\rho(u) + (\theta + c\nu_\theta)} = \frac{\rho^*}{\rho^* + (\theta + c\nu_\theta)},$$

where $\rho^* := \lim_{u\to\infty}\rho(u)$ and l'Hôpital's rule may need to find the limit;
- in all other cases when $\rho(u)$ is a non-monotonic function of u, $\underline{\aleph}(\theta)$ need to be calculated numerically. We provide numerical examples later in Section 6.

5. Estimating ruin probabilities by simulation

As many ruin problems based on our generalised risk model of (1) may lead to no closed-form results in general, we provide a numerical algorithm for efficiently simulating sample paths of the risk process X_t. Thereafter, we develop a method for estimating the ultimate ruin probability by using importance sampling via change of measure.

5.1. Numerical algorithm for exact simulation

We will first provide an efficient numerical algorithm for exact simulation (rather than considering a discrete time version of the process).

Algorithm 5.1. Given the initial condition $\left(X_0, \lambda_0, U_0\right)$, we can simulate a path of $\left\{\left(X_t, \lambda_t, U_t\right)\right\}_{t\geq 0}$ recursively by the following steps:

1. Simulate the $(k+1)$th interarrival time S_{k+1} in the point process N_t by explicitly inverting its tail distribution

$$\Pr\{S_{k+1} > s\} = \exp\left(-\int_{t_k}^{t_k+s}\lambda_{t_k^+}e^{-\delta(u-t_k)}\mathrm{d}u\right)$$

$$= \exp\left(-\frac{1-e^{-\delta s}}{\delta}\lambda_{t_k^+}\right).$$

2. Simulate the $(k+1)$th interarrival time E_{k+1} in the intensity process λ_t via

$$\Pr\{E_{k+1} > s\} = \Pr\left\{R_{k+1} > U_k + s \mid R_{k+1} > U_k\right\}$$

$$= \frac{\bar{P}(U_k+s)}{\bar{P}(U_k)},\quad R_{k+1}\sim P, \tag{14}$$

where E_{k+1} can be simulated by inversion if $\frac{\bar{P}(U_k+s)}{\bar{P}(U_k)}$ has an analytic inverse function, otherwise, E_{k+1} can be simulated by truncation; we provide a numerical example in Section 6.
3. Record the $(k+1)$th common interarrival time $I_{k+1} = \min\{E_{k+1}, S_{k+1}\}$, and the $(k+1)$th arrival time $t_{k+1} = t_k + I_{k+1}$.
4. Simulate a path of the joint process (X_t, λ_t, U_t) within the time interval $[t_k, t_k + I_{k+1})$:
   - if $\min\{E_{k+1}, S_{k+1}\} = E_{k+1}$, then, set $U_{t_{k+1}} = 0,\quad \lambda_{t_{k+1}} = \lambda_{t_k^+}e^{-\delta I_{k+1}} + Y_{k+1}$, $X_{t_{k+1}} = X_{t_k^+} + cI_{k+1}$;
   - if $\min\{E_{k+1}, S_{k+1}\} = S_{k+1}$, then, set $U_{t_{k+1}} = U_k + I_{k+1},\quad \lambda_{t_{k+1}} = \lambda_{t_k^+}e^{-\delta I_{k+1}}$, $X_{t_{k+1}} = X_{t_k^+} + cI_{k+1} - Z_{k+1}$.

5.2. Ruin probability by change of measure

Ruin is usually a rare event under the original probability measure $\mathbb{P}$ in the real world. Hence, a direct crude Monte Carlo simulation approach may not be so efficient. We extend the importance sampling methodology of Dassios and Zhao (2012) based on a suitable change of probability measure. This has a double effect:

(1) under the new probability measure the event of ruin becomes almost certain;
(2) under the new probability measure, the importance sampling estimator of the ruin probability has smaller variance (or standard error).

The general method of improving the efficiency of stochastic simulation using importance sampling in the literature can be found in Siegmund (1976), Glynn and Iglehart (1989), Glasserman (2003) and Asmussen and Glynn (2007). In particular, for ruin problems, see Asmussen (1985), Asmussen and Binswanger (1997) and Torrisi (2004).

Theorem 5.1. *If the net profit condition (3) holds under the original measure $\mathbb{P}$, the ruin probability conditional on (X_0, λ_0, U_0) can be expressed under the new measure $\tilde{\mathbb{P}}$ by*

$$\Pr\left\{\tau^* < \infty \mid X_0 = x, \lambda_0 = \lambda, U_0 = u\right\}$$

$$= e^{-\nu_0 x}e^{\theta_0\tilde{\lambda}}\hbar(u)\tilde{\mathbb{E}}\left[\Psi(X_{\tau^*_-})\frac{e^{-\theta_0\tilde{\lambda}_{\tau^*_-}}}{\hbar(U_{\tau^*_-})}\,\middle|\, X_0 = x,\right.$$

$$\left.\tilde{\lambda}_0 = \tilde{\lambda}, U_0 = u\right], \tag{15}$$

where ν_0 is defined in (ii) of Lemma 3.2, $\theta_0 := \frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}$, $\tilde{\lambda} := \phi(\nu_0)\lambda$,

$$\hbar(u) := \frac{\bar{\tilde{P}}(u)}{\bar{P}(u)}e^{c\nu_0 u},\qquad \bar{\tilde{P}}(u) := 1 - \tilde{P}(u), \tag{16}$$

$$\Psi(u) := \frac{\int_u^\infty e^{-\nu_0(z-u)}\mathrm{d}\tilde{Z}(z)}{\bar{\tilde{Z}}(u)},\qquad \bar{\tilde{Z}}(u) := 1 - \tilde{Z}(u), \tag{17}$$

with the new equivalent probability measure $\tilde{\mathbb{P}}$ defined via the Radon–Nikodym derivative (or likelihood ratio)

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\tilde{\mathbb{P}}} := e^{-\nu_0 x}e^{\theta_0\tilde{\lambda}}\hbar(u)\Psi(X_{\tau^*_-})\frac{e^{-\theta_0\tilde{\lambda}_{\tau^*_-}}}{\hbar(U_{\tau^*_-})}.$$

The associated parameter setting for the process (X_t, λ_t, U_t) under $\mathbb{P}$ transforms to the new one under $\tilde{\mathbb{P}}$ according to

$$\mathbb{P}\to\tilde{\mathbb{P}}:\quad \lambda\to\tilde{\lambda},\quad c\to\tilde{c},\quad \delta\to\tilde{\delta},$$
$$p\to\tilde{p},\quad P\to\tilde{P},\quad Z\to\tilde{Z},\quad h\to\tilde{h},$$

where $\tilde{c} = c$, $\tilde{\delta} = \delta$,

$$\tilde{p}(u) := \frac{e^{-c\nu_0 u}}{\hat{p}(c\nu_0)}p(u),\qquad \tilde{P}(u) := \int_0^u\tilde{p}(v)\mathrm{d}v, \tag{18}$$

$$\mathrm{d}\tilde{Z}(z) := \frac{e^{\nu_0 z}}{\phi(\nu_0)}\mathrm{d}Z(z),\qquad \tilde{h}(u) := \frac{e^{\frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}u}}{\phi(\nu_0)}\frac{h\left(\frac{u}{\phi(\nu_0)}\right)}{\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)}. \tag{19}$$

Proof. If we set $\theta = 0$ in Theorem 3.1 and (7) and further assume $\hbar(0) = 1$, we have the $\mathbb{P}$-martingale

$$e^{-\nu_0 X_t} e^{\frac{\phi(\nu_0)-1}{\delta}\lambda_t} \hbar(U_t), \quad t > 0, \tag{20}$$

where

$$\hbar'(u) - c\nu_0 \hbar(u) + \frac{p(u)}{\bar{P}(u)}\left[\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)\hbar(0) - \hbar(u)\right] = 0.$$

This differential equation has the solution

$$\hbar(u) = \frac{\int_u^\infty e^{-c\nu_0 v} p(v)\mathrm{d}v}{e^{-c\nu_0 u}\bar{P}(u)} \hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right). \tag{21}$$

Clearly $\hbar(u)$ is bounded, since by l'Hôpital's rule, we have

$$\lim_{u\to\infty} \frac{\int_u^\infty e^{-c\nu_0 v} p(v)\mathrm{d}v}{e^{-c\nu_0 u}\bar{P}(u)} = \lim_{u\to\infty} \frac{\frac{p(u)}{\bar{P}(u)}}{c\nu_0 + \frac{p(u)}{\bar{P}(u)}} \le 1.$$

Note that

$$\int_u^\infty e^{-c\nu_0 v} p(v)\mathrm{d}v = \hat{p}(c\nu_0)\int_u^\infty \tilde{p}(v)\mathrm{d}v = \hat{p}(c\nu_0)\bar{\tilde{P}}(u),$$

which can be rewritten (21) as

$$\hbar(u) = \hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)\hat{p}(c\nu_0)\frac{\bar{\tilde{P}}(u)}{\bar{P}(u)} e^{c\nu_0 u}.$$

Moreover, by Lemma 3.2 we have

$$\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)\hat{p}(c\nu_0) = 1$$

which can be simplified as (16).

We now carry out the change of measure via the analysis of Model-2 type (Dassios and Embrechts, 1989) generator

$$\begin{aligned}\mathcal{A}f(x,\lambda,u) &= c\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} - \delta\lambda\frac{\partial f}{\partial \lambda} \\ &\quad + \lambda\left(\int_0^x f(x-z,\lambda,u)\mathrm{d}Z(z) + \bar{Z}(x) - f(x,\lambda,u)\right) \\ &\quad + \frac{p(u)}{\bar{P}(u)}\left(\int_0^\infty f(x,\lambda+y,0)\mathrm{d}H(y) - f(x,\lambda,u)\right), \quad x > 0.\end{aligned}$$

The ruin probability under the original measure $\mathbb{P}$

$$f(x,\lambda,u) = \Pr\left\{\tau^* < \infty \mid X_0 = x, \lambda_0 = \lambda, U_0 = u\right\}$$

is the solution to the integro-differential equation $\mathcal{A}f(x,\lambda,u) = 0$. Plugging

$$f(x,\lambda,u) = e^{-\nu_0 x} e^{\frac{\phi(\nu_0)-1}{\delta}\lambda}\hbar(u)\tilde{f}(x,\lambda,u)$$

into $\mathcal{A}f(x,\lambda,u) = 0$, we have

$$\begin{aligned}0 &= c\frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{f}}{\partial u} - \delta\lambda\frac{\partial \tilde{f}}{\partial \lambda} \\ &\quad + \phi(\nu_0)\lambda\left(\int_0^x \tilde{f}(x-z,\lambda,u)\frac{e^{\nu_0 z}}{\phi(\nu_0)}\mathrm{d}Z(z) \right.\\ &\quad \left. + \frac{\bar{Z}(x)}{e^{-\nu_0 x}e^{\frac{\phi(\nu_0)-1}{\delta}\lambda}\hbar(u)\phi(\nu_0)} - \tilde{f}(x,\lambda,u)\right) \\ &\quad + \hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)\frac{\hbar(0)}{\hbar(u)}\frac{p(u)}{\bar{P}(u)} \\ &\quad \times \left(\int_0^\infty \tilde{f}(x,\lambda+y,0)\frac{e^{\frac{\phi(\nu_0)-1}{\delta}y}}{\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)}\mathrm{d}H(y) - \tilde{f}(x,\lambda,u)\right).\end{aligned}$$

Hence,

$$\begin{aligned}0 &= c\frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{f}}{\partial u} - \delta\lambda\frac{\partial \tilde{f}}{\partial \lambda} \\ &\quad + \phi(\nu_0)\lambda\left(\int_0^x \tilde{f}(x-z,\lambda,u)\frac{e^{\nu_0 z}}{\phi(\nu_0)}\mathrm{d}Z(z) \right.\\ &\quad \left. + \frac{\bar{Z}(x)}{e^{-\nu_0 x}e^{\frac{\phi(\nu_0)-1}{\delta}\lambda}\hbar(u)\phi(\nu_0)} - \tilde{f}(x,\lambda,u)\right) \\ &\quad + \frac{p(u)e^{-c\nu_0 u}}{\int_u^\infty e^{-c\nu_0 v}p(v)\mathrm{d}v} \\ &\quad \times \left(\int_0^\infty \tilde{f}(x,\lambda+y,0)\frac{e^{\frac{\phi(\nu_0)-1}{\delta}y}}{\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)}\mathrm{d}H(y) - \tilde{f}(x,\lambda,u)\right).\end{aligned}$$

Letting $\tilde{\lambda} = \phi(\nu_0)\lambda$, we have

$$\begin{aligned}0 &= c\frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{f}}{\partial u} - \delta\tilde{\lambda}\frac{\partial \tilde{f}}{\partial \tilde{\lambda}} \\ &\quad + \tilde{\lambda}\left(\int_0^x \tilde{f}(x-z,\tilde{\lambda},u)\frac{e^{\nu_0 z}}{\phi(\nu_0)}\mathrm{d}Z(z) \right.\\ &\quad \left. + \frac{\bar{Z}(x)}{e^{-\nu_0 x}e^{\frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}\tilde{\lambda}}\hbar(u)\phi(\nu_0)} - \tilde{f}(x,\tilde{\lambda},u)\right) + \frac{p(u)e^{-c\nu_0 u}}{\int_u^\infty e^{-c\nu_0 v}p(v)\mathrm{d}v} \\ &\quad \times \left(\int_0^\infty \tilde{f}\Big(x,\tilde{\lambda}+\phi(\nu_0)y,0\Big)\frac{e^{\frac{\phi(\nu_0)-1}{\delta}y}}{\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)}\mathrm{d}H(y) - \tilde{f}(x,\tilde{\lambda},u)\right).\end{aligned}$$

By the change of variable $u = \phi(\nu_0)y$, we have

$$\begin{aligned}0 &= c\frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{f}}{\partial u} - \delta\tilde{\lambda}\frac{\partial \tilde{f}}{\partial \tilde{\lambda}} \\ &\quad + \tilde{\lambda}\left(\int_0^x \tilde{f}(x-z,\tilde{\lambda},u)\frac{e^{\nu_0 z}}{\phi(\nu_0)}\mathrm{d}Z(z) \right.\\ &\quad \left. + \frac{\bar{Z}(x)}{e^{-\nu_0 x}e^{\frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}\tilde{\lambda}}\hbar(u)\phi(\nu_0)} - \tilde{f}(x,\tilde{\lambda},u)\right) + \frac{p(u)e^{-c\nu_0 u}}{\int_u^\infty e^{-c\nu_0 v}p(v)\mathrm{d}v} \\ &\quad \times \left(\int_0^\infty \tilde{f}\Big(x,\tilde{\lambda}+u,0\Big)\frac{e^{\frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}u}}{\phi(\nu_0)}\frac{h\left(\frac{u}{\phi(\nu_0)}\right)}{\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)}\mathrm{d}u - \tilde{f}(x,\tilde{\lambda},u)\right).\end{aligned}$$

Using an Esscher transform (Gerber and Shiu, 1994) on (18) and (19), we have

$$\begin{aligned}0 &= c\frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{f}}{\partial u} - \delta\tilde{\lambda}\frac{\partial \tilde{f}}{\partial \tilde{\lambda}} \\ &\quad + \tilde{\lambda}\left(\int_0^x \tilde{f}(x-z,\tilde{\lambda},u)\mathrm{d}\tilde{Z}(z) \right.\\ &\quad \left. + \frac{\bar{Z}(x)}{e^{-\nu_0 x}e^{\frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}\tilde{\lambda}}\hbar(u)\phi(\nu_0)} - \tilde{f}(x,\tilde{\lambda},u)\right) \\ &\quad + \frac{\tilde{p}(u)}{\bar{\tilde{P}}(u)}\left(\int_0^\infty \tilde{f}\Big(x,\tilde{\lambda}+u,0\Big)\tilde{h}(u)\mathrm{d}u - \tilde{f}(x,\tilde{\lambda},u)\right).\end{aligned}$$

It is easy to check that $\int_0^\infty \tilde{h}(u)\mathrm{d}u = 1$, so $\tilde{h}(u)$ is a well defined density function. Note that,

$$\bar{\tilde{Z}}(x) = \int_x^\infty \mathrm{d}\tilde{Z}(z) = \frac{\int_x^\infty e^{\nu_0 z}\mathrm{d}Z(z)}{\phi(\nu_0)}.$$

Therefore,

$$\frac{\bar{Z}(x)}{e^{-\nu_0 x} e^{\frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}\tilde{\lambda}} \hbar(u)\phi(\nu_0)} = \frac{e^{-\frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}\tilde{\lambda}}}{\hbar(u)} \frac{\int_x^\infty e^{-\nu_0(z-x)} \mathrm{d}\tilde{Z}(z)}{\bar{\tilde{Z}}(x)} \bar{\tilde{Z}}(x)$$
$$= \frac{e^{-\frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}\tilde{\lambda}}}{\hbar(u)} \Psi(x)\bar{\tilde{Z}}(x).$$

Hence, we have

$$0 = c\frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{f}}{\partial u} - \delta\tilde{\lambda}\frac{\partial \tilde{f}}{\partial \tilde{\lambda}} + \tilde{\lambda}\left(\int_0^x \tilde{f}(x-z, \tilde{\lambda}, u)\mathrm{d}\tilde{Z}(z) + \frac{e^{-\frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}\tilde{\lambda}}}{\hbar(u)} \Psi(x)\bar{\tilde{Z}}(x) - \tilde{f}(x, \tilde{\lambda}, u)\right) + \frac{\tilde{p}(u)}{\bar{\tilde{P}}(u)}\left(\int_0^\infty \tilde{f}\big(x, \tilde{\lambda}+u, 0\big)\tilde{h}(u)\mathrm{d}u - \tilde{f}(x, \tilde{\lambda}, u)\right),$$

with the solution

$$\tilde{f}(x, \tilde{\lambda}, u) = \tilde{\mathbb{E}}\left[\Psi(x_{\tau_-^*}) \frac{e^{-\frac{\phi(\nu_0)-1}{\delta\phi(\nu_0)}\tilde{\lambda}_{\tau_-^*}}}{\hbar(U_{\tau_-^*})} \mathbb{1}\{\tau^* < \infty\} \,\middle|\, X_0 = x, \tilde{\lambda}_0 = \tilde{\lambda}, U_0 = u\right].$$

We will prove in the next theorem that, if the net profit condition (3) holds under the original measure $\mathbb{P}$, then ruin occurs almost surely under the new measure $\tilde{\mathbb{P}}$. Hence, we have the ruin probability (15). □

Theorem 5.2. *If the net profit condition (3) holds under the original measure $\mathbb{P}$, then, ruin occurs almost surely under the new measure $\tilde{\mathbb{P}}$.*

Proof. Note that first-order moments under the new measure $\tilde{\mathbb{P}}$ are given by

$$\tilde{\pi}_1 := \tilde{\mathbb{E}}[R_i] = \int_0^\infty u\tilde{p}(u)\mathrm{d}u = \frac{\int_0^\infty ue^{-c\nu_0 u}p(u)\mathrm{d}u}{\hat{p}(c\nu_0)},$$

$$\tilde{\alpha}_1 := \tilde{\mathbb{E}}[Y_i] = \int_0^\infty u\tilde{h}(u)\mathrm{d}u = \phi(\nu_0)\frac{\int_0^\infty ye^{\frac{\phi(\nu_0)-1}{\delta}y}\mathrm{d}H(y)}{\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)},$$

$$\tilde{\gamma}_1 := \tilde{\mathbb{E}}[Z_j] = \int_0^\infty z\mathrm{d}\tilde{Z}(z) = \frac{\int_0^\infty ze^{\nu_0 z}\mathrm{d}Z(z)}{\phi(\nu_0)}.$$

The loss rate under the new measure $\tilde{\mathbb{P}}$ is given by

$$\frac{\tilde{\gamma}_1\tilde{\alpha}_1}{\tilde{\delta}\tilde{\pi}_1} = \frac{\hat{p}(c\nu_0)}{\delta\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)} \frac{\int_0^\infty ze^{\nu_0 z}\mathrm{d}Z(z)\int_0^\infty ye^{\frac{\phi(\nu_0)-1}{\delta}y}\mathrm{d}H(y)}{\int_0^\infty ue^{-c\nu_0 u}p(u)\mathrm{d}u}.$$

From (5), we have

$$f_0(\nu) = \hat{p}(c\nu)\hat{h}\left(-\frac{\phi(\nu)-1}{\delta}\right),$$

$$f_0'(\nu) = \hat{p}(c\nu)\frac{\int_0^\infty ze^{\nu z}\mathrm{d}Z(z)}{\delta}\int_0^\infty ye^{\frac{\phi(\nu)-1}{\delta}y}\mathrm{d}H(y) - c\hat{h}\left(-\frac{\phi(\nu)-1}{\delta}\right)\int_0^\infty ue^{-c\nu u}p(u)\mathrm{d}u.$$

From the net profit condition (3), we have

$$f_0'(\nu)\big|_{\nu=0} = -c\pi_1 + \frac{\gamma_1\alpha_1}{\delta} < 0.$$

This is due to the convexity of $f(\nu)$ as proved in Lemma 3.2, i.e. $f''(\nu) > 0$. Recall that υ_0 is the unique positive solution to (4) for $\theta = 0$ (see Fig. 2) and we have $f_0'(\nu)\big|_{\nu=\nu_0>0} > 0$. Then,

$$\hat{p}(c\nu_0)\frac{\int_0^\infty ze^{\nu_0 z}\mathrm{d}Z(z)}{\delta}\int_0^\infty ye^{\frac{\phi(\nu_0)-1}{\delta}y}\mathrm{d}H(y) > c\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)\int_0^\infty ue^{-c\nu_0 u}p(u)\mathrm{d}u$$

which can be rewritten as

$$\frac{\hat{p}(c\nu_0)\int_0^\infty ze^{\nu_0 z}\mathrm{d}Z(z)\int_0^\infty ye^{\frac{\phi(\nu_0)-1}{\delta}y}\mathrm{d}H(y)}{\delta\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)\int_0^\infty ue^{-c\nu_0 u}p(u)\mathrm{d}u} > c,$$

i.e.

$$\frac{\tilde{\gamma}_1\tilde{\alpha}_1}{\tilde{\delta}\tilde{\pi}_1} > \tilde{c}.$$

Hence, the expected loss rate exceeds the expected premium rate, and ruin is almost certain to happen under the new measure $\tilde{\mathbb{P}}$. □

We now analyse the efficiency of our simulation scheme based on importance sampling developed in Theorem 5.1. In the following Corollary 5.1, we prove that, for a relatively large initial reserve, the new variance of the estimator for the ultimate ruin probability based on importance sampling in Theorem 5.1 is less than the variance of the estimator based on the crude simulation of Algorithm 5.1 under the original probability measure.

Corollary 5.1. *For any initial reserve $x > \underline{x}$ where*

$$\underline{x} := \frac{1}{\nu_0}\left[\theta_0\tilde{\lambda} + \ln\frac{\aleph(u,0)}{\underline{\aleph}(0)}\right],$$

and $\aleph(u,0)$, $\underline{\aleph}(0)$ are defined by (10), we have

$$\mathbb{V} > \tilde{\mathbb{V}},$$

where $\mathbb{V}$ is the variance of the estimator for the ultimate ruin probability based on the crude simulation under the original measure $\mathbb{P}$, and $\tilde{\mathbb{V}}$ is the variance based on the importance sampling procedure under the new measure $\tilde{\mathbb{P}}$, i.e.

$$\mathbb{V} := \mathrm{Var}\Big[\mathbb{1}\{\tau^* < \infty \mid X_0 = x, \lambda_0 = \lambda, U_0 = u\}\Big],$$

$$\tilde{\mathbb{V}} := \widetilde{\mathrm{Var}}\left[e^{-\nu_0 x}e^{\theta_0\tilde{\lambda}}\hbar(u) \times \Psi(X_{\tau_-^*})\frac{e^{-\theta_0\tilde{\lambda}_{\tau_-^*}}}{\hbar(U_{\tau_-^*})} \,\middle|\, X_0 = x, \tilde{\lambda}_0 = \tilde{\lambda}, U_0 = u\right].$$

Proof. We have that $\mathbb{V} = \psi(1-\psi)$ where $\psi := \Pr\{\tau^* < \infty \mid X_0 = x, \lambda_0 = \lambda, U_0 = u\}$ and

$$\tilde{\mathbb{V}} = \tilde{\mathbb{E}}\left[\left(e^{-\nu_0 x}e^{\theta_0\tilde{\lambda}}\hbar(u) \times \Psi(X_{\tau_-^*})\frac{e^{-\theta_0\tilde{\lambda}_{\tau_-^*}}}{\hbar(U_{\tau_-^*})}\right)^2 \,\middle|\, X_0 = x, \tilde{\lambda}_0 = \tilde{\lambda}, U_0 = u\right] - \psi^2.$$

Based on $\hbar(u)$ as specified in (21) and further discussions on lower bounds in Remark 4.1, we have

$$\hbar(u) = \aleph(u,0)\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right) \geq \underline{\aleph}(0)\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right).$$

Moreover, note that $\Psi(X_{\tau_-^*}) < 1$ always holds, so we have

$$\Psi(X_{\tau_-^*})\frac{e^{-\theta_0\tilde{\lambda}_{\tau_-^*}}}{\hbar(U_{\tau_-^*})} \leq \frac{1}{\underline{\aleph}(0)\hat{h}\left(-\frac{\phi(\nu_0)-1}{\delta}\right)}.$$

Given $\hbar(u)$ from (21) and $\aleph(u,0)$ from (10), it is clear that, if x is large enough, more precisely, if $x > \underline{x}$, we have

$$e^{-\nu_0 x}e^{\theta_0\tilde{\lambda}}\hbar(u) \times \Psi(X_{\tau_-^*})\frac{e^{-\theta_0\tilde{\lambda}_{\tau_-^*}}}{\hbar(U_{\tau_-^*})} < 1.$$

Therefore,

$$\begin{aligned}\tilde{\mathbb{V}} - \mathbb{V} &= \tilde{\mathbb{E}}\left[\left(e^{-\nu_0 x}e^{\theta_0\tilde{\lambda}}\hbar(u) \times \Psi(X_{\tau_-^*})\frac{e^{-\theta_0\tilde{\lambda}_{\tau_-^*}}}{\hbar(U_{\tau_-^*})}\right)^2 \middle| X_0 = x,\right.\\ &\qquad \left.\tilde{\lambda}_0 = \tilde{\lambda}, U_0 = u\right] - \psi \\ &< \tilde{\mathbb{E}}\left[e^{-\nu_0 x}e^{\theta_0\tilde{\lambda}}\hbar(u) \times \Psi(X_{\tau_-^*})\frac{e^{-\theta_0\tilde{\lambda}_{\tau_-^*}}}{\hbar(U_{\tau_-^*})} \middle| X_0 = x,\right.\\ &\qquad \left.\tilde{\lambda}_0 = \tilde{\lambda}, U_0 = u\right] - \psi \\ &= \psi - \psi = 0\end{aligned}$$

and $\tilde{\mathbb{V}} < \mathbb{V}$. □

Remark 5.1. If $\rho(u)$ is a non-decreasing function of u, then, using (13) and Lemma 3.2, we have $\underline{\aleph}(0) = \hat{p}(c\nu_0)$, and an explicit lower bound for x,

$$\underline{x} = \frac{1}{\nu_0}\left[\theta_0\tilde{\lambda} + \ln\hbar(u)\right].$$

If we further assume $u = 0$, we simply have $\underline{x} = \frac{\theta_0}{\nu_0}\tilde{\lambda}$.

Remark 5.2. In fact, Theorem 5.1 combined with Corollary 5.1 tells us that,

$$\tilde{\mathbb{V}} = \mathbb{V} \times O\left(e^{-\nu_0 x}\right),$$

which demonstrates the efficiency of the importance sampling approach for a large initial reserve x. In practice, the initial reserve is usually large, so the condition $x > \underline{x}$ is not a serious restriction. Further improvements to the efficiency of our algorithm can be a subject of future research.

6. Numerical implementation

For numerical implementation, we assume explicitly that, under the measure $\mathbb{P}$, the claim sizes $\{Z_j\}_{j=1,2,\ldots}$ and jump sizes $\{Y_i\}_{i=1,2,\ldots}$ follow exponential distributions, and the interarrival times $\{R_i\}_{i=1,2,\ldots}$ follow an inverse Gaussian distribution, say,

$$Z \sim \mathrm{Exp}(\gamma), \qquad H \sim \mathrm{Exp}(\alpha),$$

$$P \sim \mathrm{IG}\left(\mu_{IG} = \frac{a}{b}, \lambda_{IG} = a^2\right),$$

where α, γ, a, b are all positive constants. We will now explain how to implement our model step by step.

Distribution of claim sizes Z. If the claim sizes are exponentially distributed with parameter γ under the measure $\mathbb{P}$, we have $\gamma_1 = 1/\gamma$, $\phi(\nu_0) = \frac{\gamma}{\gamma-\nu_0}$, and $\Psi(u)$ of (17) can be simplified as $\Psi(u) = \frac{\gamma-\nu_0}{\gamma}$ and is independent of X_t. Hence, we do not need to record $X_{\tau_-^*}$ during the simulation for this special case. By transformation (19), we have $\mathrm{d}\tilde{Z}(z) = (\gamma-\nu_0)e^{-(\gamma-\nu_0)z}\mathrm{d}z$, which implies that $\tilde{Z} \sim \mathrm{Exp}(\gamma-\nu_0)$, $\gamma > \nu_0$ under the measure $\tilde{\mathbb{P}}$.

Distribution of interarrival times P. The distributional properties of inverse Gaussian distribution have been well documented in Chhikara and Folks (1989). If P follows an inverse Gaussian distribution, say, $P \sim \mathrm{IG}\left(\mu_{IG} = \frac{a}{b}, \lambda_{IG} = a^2\right)$ with mean $\pi_1 = \mu_{IG} = a/b$ and the shape parameter $\lambda_{IG} = a^2$, then, we have the density

$$p(u) = \frac{a}{\sqrt{2\pi u^3}}e^{-\frac{(a-bu)^2}{2u}},$$

the Laplace transform

$$\hat{p}(v) = e^{-\left(\sqrt{2v+b^2}-b\right)a},$$

and the cumulative distribution function

$$P(u) = \Phi\left(\frac{bu-a}{\sqrt{u}}\right) + e^{2ab}\Phi\left(-\frac{bu+a}{\sqrt{u}}\right), \tag{22}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

To calculate an upper bound for the ruin probability as given by Theorem 4.1 and further based on Remark 4.1, since the failure rate, $\rho(u)$, is a non-monotonic function (Chhikara and Folks, 1977), we need to calculate $\underline{\aleph}(\theta)$ numerically. The key function $\aleph(u,\theta)$ defined by (10) can be calculated explicitly by Eq. (23) given in Box I.

For the Monte Carlo simulation via Algorithm 5.1, we note that by (14), we have

$$\Pr\{E_{k+1} > s\} = \frac{\bar{P}(U_k+s)}{\bar{P}(U_k)},$$

where

$$\bar{P}(u) = \Phi\left(-\frac{bu-a}{\sqrt{u}}\right) - e^{2ab}\Phi\left(-\frac{bu+a}{\sqrt{u}}\right).$$

However, the analytic inverse function for s does not exist, so we have to simulate E_{k+1} by truncating the inverse Gaussian distribution. An efficient simulation algorithm of the inverse Gaussian distribution can be found in Michael et al. (1976). By transformation (18), it remains an inverse Gaussian distribution, since the density under the measure $\tilde{\mathbb{P}}$ is given by

$$\tilde{p}(u) = \frac{a}{\sqrt{2\pi u^3}}e^{-\frac{\left(a-\sqrt{2c\nu_0+b^2}u\right)^2}{2u}},$$

then, $\tilde{P} \sim \mathrm{IG}\left(\tilde{\mu}_{IG} = \frac{\tilde{a}}{\tilde{b}}, \tilde{\lambda}_{IG} = \tilde{a}^2\right)$ where $\tilde{a} = a$, $\tilde{b} = \sqrt{2c\nu_0+b^2}$, and

$$\tilde{P}(u) = \Phi\left(\frac{\tilde{b}u-\tilde{a}}{\sqrt{u}}\right) + e^{2\tilde{a}\tilde{b}}\Phi\left(-\frac{\tilde{b}u+\tilde{a}}{\sqrt{u}}\right). \tag{24}$$

Distribution of Jump sizes H. If we further assume $H \sim \mathrm{Exp}(\alpha)$, then, we have $\alpha_1 = 1/\alpha$, $\hat{h}(\nu) = \frac{\alpha}{\alpha+\nu}$ and

$$\hbar(u) = \frac{1-\tilde{P}(u)}{1-P(u)}e^{c\nu_0 u},$$

$$\aleph(u,\theta)=e^{(\theta+c\nu_\theta)u-a\left(\sqrt{b^2+2(\theta+c\nu_\theta)}-b\right)}\frac{\Phi\left(-\frac{\sqrt{b^2+2(\theta+c\nu_\theta)}u-a}{\sqrt{u}}\right)-e^{2a\sqrt{b^2+2(\theta+c\nu_\theta)}}\Phi\left(-\frac{\sqrt{b^2+2(\theta+c\nu_\theta)}u+a}{\sqrt{u}}\right)}{\Phi\left(-\frac{bu-a}{\sqrt{u}}\right)-e^{2ab}\Phi\left(-\frac{bu+a}{\sqrt{u}}\right)} \qquad (23)$$

Box I.

where $P(u)$ and $\tilde{P}(u)$ are specified by (22) and (24) respectively. By transformation (19), we have

$$\tilde{h}(u)=\left(\frac{\alpha\delta+\phi(\nu_0)-1}{\delta\phi(\nu_0)}\right)e^{-\left(\frac{\alpha\delta+\phi(\nu_0)-1}{\delta\phi(\nu_0)}\right)u}.$$

Hence, $\tilde{H}\sim \mathrm{Exp}\left(\frac{\alpha\delta+\phi(\nu_0)-1}{\delta\phi(\nu_0)}\right)$ under the measure $\tilde{\mathbb{P}}$.

Note that, the function $f_0(\nu)$ as defined by (5) is given by

$$f_0(\nu)=e^{-\left(\sqrt{2c\nu+b^2}-b\right)a}\times\frac{\alpha}{\alpha-\frac{\frac{\gamma}{\gamma-\nu}-1}{\delta}},\quad \nu\in\left[0,\ \frac{\alpha\delta}{1+\alpha\delta}\gamma\right).$$

The key parameter ν_0 can be found numerically (see Fig. 2). From (3), the net profit condition is $c>\frac{b}{\delta a\gamma\alpha}$. We set the following parameter values

$$(\delta,c;\lambda_0,X_0,U_0;a,b,\alpha,\gamma)=(2,8;1.5,10,0;0.5,5,2,0.5).$$

We can now estimate the ruin probability $\Pr\{\tau^*\le T\mid X_0,\lambda_0,U_0\}$ for any fixed time T based on Algorithm 5.1 using crude Monte Carlo simulation[1] with 10,000 replications, and the estimated ruin probabilities for different times T. The corresponding standard errors and running (CPU) times are given by Table 1 respectively. As each path is independently generated, it is obvious that, the standard error is $\sqrt{\frac{\psi(1-\psi)}{n}}$ where ψ is the associated true ruin probability and n is the total number of replications.

It is not so efficient (in fact, impossible in the strict sense) to estimate the ultimate ruin probability based on crude Monte Carlo simulation under this original probability measure $\mathbb{P}$, as we need to set the time T sufficiently large in order to approximate the infinite horizon case. Ruin has a relatively small probability, so most of the simulated samples are thrown away.

Alternatively, we can change the measure from $\mathbb{P}$ to $\tilde{\mathbb{P}}$ according to Theorem 5.1, and the transformed parameters under $\tilde{\mathbb{P}}$ are given by

$$(\tilde{\delta},\tilde{c};\tilde{\lambda}_0,X_0,U_0;\tilde{a},\tilde{b},\tilde{\alpha},\tilde{\gamma})$$
$$=(2,8;2.20,10,0;0.5,5.25,1.52,0.34),\qquad \nu_0\approx 0.1594,$$

where we find that all replications lead to ruin occurring before time $T=200$ and 91.70% of the replications before time $T=20$, see the second column in Table 2. By using the formula (15), we estimate the ultimate ruin probability as $\Pr\{\tau^*<\infty\mid X_0,\lambda_0,U_0\}\approx$ 10.29%. Note that, $\lambda_{\tau^*_-}=\lambda_{\tau^*}$ as λ_t is continuous at τ^*.

The standard error is used for measuring the error, and it is estimated by the sample standard deviation of the simulation output divided by the square root of the number of trials. Comparing Tables 1 and 2, we see that the simulation is much more efficient under $\tilde{\mathbb{P}}$ than the one under $\mathbb{P}$. The standard error is substantially reduced by about 14 times under $\tilde{\mathbb{P}}$ in average. Moreover, the computing speed is much faster under $\tilde{\mathbb{P}}$; the simulation for the case $T=200$ in Table 2 needed 19 s, whereas the simulation for an even shorter period of $T=100$ in Table 1 needed 258 s. This demonstrates the points we made under Corollary 5.1 Remark 5.2.

[1] All simulations in this paper are based on MatLab on a desktop PC with Intel Core i7-3770 CPU@3.40 GHz processor, 8.00 GB RAM, 64-bit Operating System Windows 7.

Table 1
Ruin probability $\Pr\{\tau^*\le T\mid X_0,\lambda_0,U_0\}$ estimated based on crude Monte Carlo simulation of 10,000 replications under the measure $\mathbb{P}$.

Time T	Ruin probability	Standard error ($\times 10^{-4}$)	CPU time (s)
10	8.45%	27.82	29.59
20	9.37%	29.14	54.18
30	9.22%	28.93	93.02
40	9.32%	29.07	104.71
50	8.47%	27.84	128.76
60	9.06%	28.71	162.01
70	9.27%	29.00	191.74
80	9.01%	28.63	204.56
90	8.94%	28.53	244.25
100	9.01%	28.63	257.95

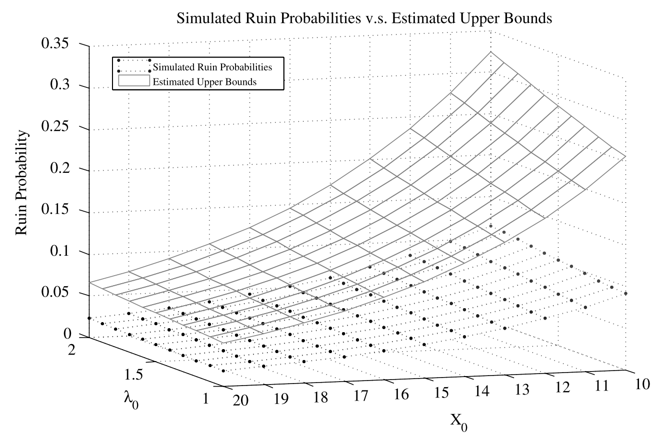

Fig. 3. Simulated ruin probabilities v.s. estimated upper bounds.

We also provide an estimated upper bound for the ultimate ruin probability based on Theorem 4.1 by letting the initial conditions X_0,λ_0 free and keeping other parameters the same, i.e. $(\delta,c;U_0;a,b,\alpha,\gamma)=(2,8;0;0.5,5,2,0.5)$. We can calculate $\aleph(U_0,0)=\underline{\aleph}(0)=\aleph(0,0)=0.8830$ based on (23), and then the upper bound can be derived by

$$\Pr\left\{\tau^*<\infty\mid X_0,\lambda_0\right\}\le e^{-0.1594X_0}e^{0.2340\lambda_0}.$$

It is plotted in Fig. 3 for $(X_0,\lambda_0)\in[10,20]\times[1,2]$ against the associated estimated ruin probabilities by simulation under the measure $\tilde{\mathbb{P}}$. Higher initial intensity λ_0 corresponds to higher ruin probability, as it signifies a higher rate of incidence of claims initially. The underlying numerical results are represented in Tables 3 and 4. This is provided as a very quick alternative (without simulation) to the other two methods.

In particular, we are interested in exploring how the distribution P of the renewal interarrival times affects the ultimate ruin probability. The inverse Gaussian distribution has two parameters: the mean $\mu_{IG}=a/b$ and the shape parameter $\lambda_{IG}=a^2$ specified at the beginning of Section 6. Note that λ_{IG} also controls the variance as the variance is μ_{IG}^3/λ_{IG}. It is obvious that the lower the mean μ_{IG} the higher the ruin probability. However, it is unclear how the shape parameter λ_{IG} affects the ruin probability. So, we fix the same level for the mean $\mu_{IG}=a/b=0.1$ as before and at the same time vary λ_{IG}; all other parameters of course are kept constant, i.e. $(\delta,c;\lambda_0,\mu_{IG},X_0,U_0;\mu_{IG},\alpha,\gamma)=$

Table 2
Ultimate ruin probability $\Pr\{\tau^* < \infty \mid X_0, \lambda_0, U_0\}$ estimated based on Monte Carlo simulation of 10,000 replications under the measure $\tilde{\mathbb{P}}$.

Time T	Ruin probability under $\tilde{\mathbb{P}}$	Ultimate ruin probability	Standard error $(\times 10^{-4})$	CPU time (s)
20	91.70%	10.34%	2.09	17.00
40	97.75%	10.29%	2.07	19.38
60	99.40%	10.28%	2.09	23.43
80	99.72%	10.29%	2.07	20.61
100	99.90%	10.25%	2.10	21.23
120	99.96%	10.25%	2.08	20.56
140	99.98%	10.29%	2.07	20.00
160	99.96%	10.29%	2.10	21.04
180	100.00%	10.31%	2.08	20.16
200	100.00%	$\boxed{10.29\%}$	2.10	19.36

Table 3
Ultimate ruin probability $\Pr\{\tau^* < \infty \mid X_0, \lambda_0\}$ (%) estimated based on Monte Carlo simulation of 10,000 replications under the measure $\tilde{\mathbb{P}}$.

$X_0 \backslash \lambda_0$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
10	9.17	9.37	9.55	9.84	10.07	10.29	10.55	10.77	11.01	11.28	11.49
11	7.74	7.95	8.15	8.35	8.54	8.74	8.93	9.15	9.34	9.60	9.77
12	6.61	6.75	6.94	7.09	7.25	7.43	7.59	7.78	7.97	8.13	8.33
13	5.61	5.73	5.88	6.03	6.18	6.30	6.45	6.61	6.77	6.90	7.07
14	4.77	4.88	5.00	5.12	5.25	5.37	5.51	5.64	5.74	5.88	6.01
15	4.05	4.18	4.25	4.35	4.47	4.56	4.67	4.79	4.89	5.02	5.12
16	3.45	3.54	3.62	3.71	3.80	3.89	3.98	4.08	4.16	4.28	4.39
17	2.94	3.01	3.08	3.16	3.24	3.32	3.39	3.47	3.55	3.62	3.72
18	2.50	2.57	2.62	2.69	2.75	2.81	2.89	2.95	3.03	3.08	3.17
19	2.14	2.19	2.23	2.29	2.35	2.40	2.46	2.51	2.58	2.64	2.69
20	1.82	1.86	1.90	1.96	1.99	2.04	2.09	2.14	2.19	2.24	2.29

Table 4
The estimated upper bounds for the ultimate ruin probability $\Pr\{\tau^* < \infty \mid X_0, \lambda_0\}$ (%).

$X_0 \backslash \lambda_0$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
10	25.67	26.28	26.90	27.53	28.19	28.85	29.54	30.24	30.95	31.68	32.44
11	21.89	22.40	22.94	23.48	24.03	24.60	25.19	25.78	26.39	27.02	27.66
12	18.66	19.10	19.56	20.02	20.49	20.98	21.47	21.98	22.50	23.04	23.58
13	15.91	16.29	16.67	17.07	17.47	17.89	18.31	18.74	19.19	19.64	20.11
14	13.57	13.89	14.22	14.55	14.90	15.25	15.61	15.98	16.36	16.75	17.14
15	11.57	11.84	12.12	12.41	12.70	13.00	13.31	13.63	13.95	14.28	14.62
16	9.86	10.10	10.34	10.58	10.83	11.09	11.35	11.62	11.89	12.18	12.46
17	8.41	8.61	8.81	9.02	9.24	9.45	9.68	9.91	10.14	10.38	10.63
18	7.17	7.34	7.52	7.69	7.88	8.06	8.25	8.45	8.65	8.85	9.06
19	6.12	6.26	6.41	6.56	6.72	6.87	7.04	7.20	7.37	7.55	7.73
20	5.21	5.34	5.46	5.59	5.73	5.86	6.00	6.14	6.29	6.44	6.59

Table 5
Ultimate ruin probability $\Pr\{\tau^* < \infty \mid X_0, \lambda_0, U_0\}$ for $\mu_{IG} = 0.1$ with different λ_{IG}, estimated based on Monte Carlo simulation of 10,000 replications under the measure $\tilde{\mathbb{P}}$.

λ_{IG}	Ruin probability under $\tilde{\mathbb{P}}$	Ultimate ruin probability	Standard error $(\times 10^{-4})$	CPU time (s)
0.1	100.00%	11.71%	2.5369	22.15
0.2	100.00%	10.52%	2.1741	19.22
0.3	100.00%	10.13%	2.0316	23.46
0.4	100.00%	9.94%	1.9770	17.96
0.5	100.00%	9.80%	1.9162	18.22
0.6	100.00%	9.71%	1.8614	17.82
0.7	100.00%	9.67%	1.8603	17.52
0.8	100.00%	9.61%	1.8375	16.97
0.9	100.00%	9.57%	1.8399	16.94
1	100.00%	9.56%	1.8093	17.53
2	100.00%	9.47%	1.7774	16.16
3	100.00%	9.45%	1.7677	16.01
4	100.00%	9.44%	1.7435	16.47
5	100.00%	9.45%	1.7593	16.72
6	100.00%	9.42%	1.7636	15.71
7	100.00%	9.42%	1.7184	15.69
8	100.00%	9.42%	1.7576	15.54
9	100.00%	9.44%	1.7348	15.63
10	100.00%	9.43%	1.7557	15.91

(2, 8; 1.5, 10, 0; 0.1, 2, 0.5). The results of this experiment with different values for λ_{IG} are represented in Table 5, and each estimated value is based on 10,000 replications under the measure $\tilde{\mathbb{P}}$ within the time $T = 200$. The second column tells that all the replications simulated under $\tilde{\mathbb{P}}$ had ruin occurring before time $T = 200$ which also confirms Theorem 5.2. The third column shows that the estimated ultimate ruin probabilities have some negative relationship with λ_{IG} (i.e. positive relationship with variance of renewal interarrival times). However, sensitivity to this parameter decreases, as λ_{IG} increases.

Acknowledgements

The authors would like to thank all reviewers for extremely helpful and constructive comments and suggestions. The corresponding author Hongbiao Zhao acknowledges the financial support from the National Natural Science Foundation of China (#71401147) and the research fund provided by Xiamen University.

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